Airline Networks, Traffic Densities, and Value of Links

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Abstract

In airline networks, a link creates profits for its carrier in conjunction with the carrier’s other links. In other words, a link has “network” value. One prominent mechanism behind this network value is a “hubbing effect:” adding one single link to a hub creates many connecting routes. This paper studies another less explored mechanism of network value. It relies on the observation that passengers generally prefer a higher flight frequency (mainly because it provides more flexible options for travel times). When a carrier adds a link, the created connecting routes increase the traffic densities on adjacent links. As the carrier adjusts flight frequencies to meet these higher densities, there creates a positive effect on demand. By estimating a structural model that incorporates this mechanism, I am able to quantify the density effect. It is found to be several times larger than the hubbing effect. Furthermore, the model implies that the competitive impact of an airline entry (i.e., adding a new link) goes greatly beyond the local city-pair market where the entry happens.

1 Introduction

One of the reasons that make the airline industry particularly interesting is that it operates on a network structure, where cities are connected by links (i.e., direct services). This, together with the economic importance of the industry, has attracted many researchers in economics, operations research, as well as network science.¹ For airline carriers, one important consideration in designing their networks is the efficient placement of links. For example, adding one link to a hub creates not only a non-stop route but also many one-stop routes connecting to the cities that are already linked to the hub. This “hubbing effect” is probably the most straightforward explanation that

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has been put forth for the hub-spoke configuration (Hendricks et al., 1995, 1999; Gastner and Newman, 2006).

An implication of the hubbing effect is that a link creates value for the carrier not in isolation but in conjunction with the rest of the carrier’s network – the so-called network value. This idea has led to fruitful research in industrial organization. Most notable examples include Mayer and Sinai (2003), which uses the externalities associated with the hubbing effect to explain airport congestions, and Goolsbee and Syverson (2008), which uses a carrier’s presences at two cities to measure the level of “threat of entry” from the carrier to establish a link between these two cities.

In this paper, I study a different mechanism through which a link can create network value. When a carrier adds a new link, it creates connecting routes with adjacent links. As a result, the traffic densities on these adjacent links will increase, which typically imply higher flight frequencies. Because passengers are known to value higher flight frequencies, this creates a positive effect on demand. Moreover, the increases in demand imply further increases in densities, so the impact re-inforces itself. The impact can also spread to farther links in the network. For a concrete example, consider a scenario where United Airlines removes its link between Houston and Chicago (the scenario of adding the link entails exactly the opposite effects). The passengers that have been traveling on, say, the Houston-Chicago-Buffalo route can no longer do so. This reduces the traffic density, consequently also the flight frequency, on United’s Chicago-Buffalo link. This lower frequency will negatively affect the passengers who rely on United’s service between Chicago and Buffalo. First, it becomes more difficult for a passenger to find a flight that departs or arrives at her desired times of travel (i.e., schedule delays, as coined by Douglas and Miller, 1974). Second, it is more difficult to catch another flight in the event of unexpected delays or cancellations. Third, for transferring passengers, there is the additional inconvenience of a longer wait time at Chicago or Buffalo. As a result, the demand on multiple routes will fall, accompanied by falls in densities too.

Conceptually, this mechanism as described above relies on the interplay of two channels: (i) flight frequencies are adjusted to traffic densities and (ii) passengers value higher frequencies. The literature has documented both channels, though separately for the most part.\(^2\) Perhaps the only exception is the theory work by Brueckner (2004). He considered a monopoly carrier’s choice of networks among 3 cities when both channels are present. He shows that the carrier has a tendency to use hub-spoke networks. The literature has also proposed alternative explanations for hub-spoke structure. A prominent one is that a dominant presence at an airport offers increased market power (Borenstein, 1989; Ciliberto and Williams, 2010). Another one is that higher traffic densities reduce marginal costs (Caves et al. 1984; Brueckner and Spiller, 1994). In contrast, I focus on how higher densities induce a demand-side benefit, and particularly how this benefit compares with the hubbing effect.\(^3\)

\(^2\)See Section 2.2 for some references.

\(^3\)In this paper, I use “the hubbing effect” in a narrower sense to only indicate the benefits of creating new connecting routes. Practically, both the alternative channels mentioned here (airport dominance and economies of
This paper quantifies the density effect through a structural model. The model, presented in Section 2, allows me to compute the exact value of a link through the counterfactuals of adding or removing that link, while accounting for the price and frequency adjustments by the carriers over the entire U.S. network. In the model, passengers traveling between two cities (i.e., a city-pair market) choose from competing routes. A route connects two cities using one or more consecutive links. In this regard, the model follows the previous discrete-choice studies (Berry et al., 2006; Peters, 2006; Armantier and Richard, 2008; Berry and Jia, 2010). However, I incorporate the density effect by allowing the density on a link, which equals to the aggregate demand across the routes that use this link, to affect the demand for these routes. This implies a circular role of density and I model it by a fixed point. As a result, the city-pair markets are no longer separate as in previous studies but tied together through the network structure. Each carrier sets the prices on its routes all together over the entire network in a Nash-Bertrand equilibrium. The flight frequency choices are implicitly accounted for by the density levels. This simplifies the carrier’s problem and makes the model tractable.

In Section 3 and 4, the model parameters are estimated from the DB1B data. The estimate of the passenger utility for density turns out to be in line with the literature’s estimates on channel (i) and (ii). The estimated marginal costs are in line with the operational costs reported by the airlines, which is not the case if the model excludes the density effect.

In Section 5, I compute the value of each link. This amounts to calculating the profit change for the carrier when a link is added (back) to its network. The value is decomposed to be attributed to different mechanisms, including the hubbing and density effect. On average, the density effect is found to be several times larger. In particular, due to the density effect, there is an overall increase in the profits on the carrier’s existing routes when a new link is added. This would not be the case in the models considered in previous studies, which did not account for possible flight frequency changes in counterfactuals. If anything, these models would predict decreases in the profits on the carrier’s existing routes due to cannibalization.

I also examine the dis-value of a link to competing carriers. Studies of airline entry naturally focus their attention on the city-pair market where the entry (i.e., addition of a link) happens. However, it seems important to also look at the network-wide impact, especially in the presence of the density effect. Indeed, I find that on average, rival carriers suffer the majority of their profit decreases on the routes outside the market of entry. In general, these counterfactuals suggest that in a networked market, the repercussions of an event can well exceed its local impact.
Notes: The nodes represent the cities. The solid lines represent the links of carrier 1. The dashed lines represent the links of carrier 2.

Figure 1: An Example of Airline Network.

2 Model

2.1 Links and routes

In airline networks, a link represents the direct service provided by a carrier between two cities. Accordingly, a link is identified by a carrier and a pair of cities. In this paper, I denote a link generically by \( g \). Links are considered undirected as it is rare for a carrier to serve only one direction between two cities. Figure 1 illustrates a simple network with two carriers serving seven cities. The solid lines represent the links of carrier 1, while the dashed lines represent carrier 2. For the passengers traveling between city 1 and 2, they can choose to take either carrier’s non-stop route or carrier 1’s one-stop route that makes a connection at city 1. More generally, a route is made of one or more links (commonly known as the segments of the route) so that it can carry passengers between two cities. A non-stop route consists of a single link, while a one-stop route consists of two links. I treat routes as undirected too.\(^4\)

In general, the segments of a route can be operated by different carriers if there is a code-share agreement between them. However, such routes carried only slightly more than 2% of the passengers in the data. For simplicity, I do not consider these routes in this paper. As a result, each route belongs to exactly one carrier. It is equally rare to see passengers flying routes with more than one stop, so I will not consider these routes either. Conceptually, it is not difficult to extend my model to include code-shared or multi-stop routes.

Suppose that the demand for each route is given, then one can calculate the passenger density (also known as hubbing effects). My model can incorporate these two channels (see Sec. 2.3 and 4.1).

\(^4\)In some previous discrete choice studies, routes are defined as round-trip itineraries. While conceptually it is not difficult to apply such definitions in my model, the greatly increased number of products will render the computation much more difficult.
on any link simply as the sum of the demand across the routes for which that route is used as a segment. Formally, let \( n \) be the total number of routes across all carriers and let \( D_j \) be the demand for route \( j = 1, \ldots, n \). Then the density on link \( g \) is given by

\[
F_g(D) = \sum_{j=1}^{n} D_j \cdot 1\{g \in j\},
\]

(2.1)

where \( g \in j \) denotes that link \( g \) is a segment of \( j \). Notice that since both link and route are carrier-specific, if \( g \) does not belong to the carrier that operates \( j \), we have \( g \not\in j \). As an example, consider Figure 1 again; the density on carrier 2’s link between city 1 and 2 should be the sum of the demand across two routes: (i) carrier 2’s non-stop route between city 1 and 2, and (ii) carrier 2’s one-stop route between city 1 and 3.

### 2.2 Passenger demand

Following previous discrete-choice studies, passenger demand is generated at the city-pair level. Fix a city-pair market \( m \). The consumers in this market want to travel between the two cities. The set of products in this market consists of the non-stop and one-stop routes that can carry passengers between the two cities. These routes are differentiated by travel length, price, flight frequency, among other things. The utility of taking route \( j \) for individual \( i \) is specified as

\[
u_{ij} = \beta' x_j - \alpha p_j + f_j + \xi_j + \varepsilon_{ij}.
\]

(2.2)

The first term \( x_j \) is a vector of observed characteristics (such as the carrier dummies, route length, and whether \( j \) is a non-stop route). \( p_j \) is the price on route \( j \). \( \xi_j \) is a route-specific component that captures the econometrician-unobserved characteristics of the route. \( f_j \) measures the densities on the segments of route \( j \), and it implicitly captures the passenger’s utility from flight frequency. For a non-stop route, I specify that

\[
f_j = \theta_1 \log F_g(D), \quad \text{where } j = \{g\}.
\]

(2.3)

Here \( g \) denotes the only segment of route \( j \) and \( F_g \) is the density on link \( g \) as defined in (2.1). The logarithm specification is motivated by the diminishing effect of frequency: the average interval between two consecutive flights decreases substantially when the frequency increases from 1 to 2 flights per day, but much less so when the frequency increases from 10 to 11 per day. The log specification was also used in Peters (2006) and Hess and Polak (2005). The latter also argues that the log specification gives superior fit than other non-linear specifications.\(^5\)

Next, consider the specification of \( f_j \) for a one-stop route. In general, the higher the frequency on either segment is, the easier it becomes to schedule connecting flights for the route. So \( f_j \) needs

\[^5\text{Alternatively, the log specification can be derived from a nested logit model in which passengers first choose a route then a flight. In this case, the “utility” for a route will increase logarithmically with the number of its flights, and } \theta_1 \text{ is equivalent to the nesting parameter (or logsum parameter).}\]
to account for the densities on both segments. One possibility is to use the geometric mean:

\[ f_j = \theta_2 \log \sqrt{F_g(D) \cdot F_{g'}(D)}, \quad \text{where } j = \{g, g'\}. \] (2.4)

Here \( g \) and \( g' \) denote the two segments of \( j \). With this specification, the lower-density segment matters more than the higher-density segment. This is intuitive because the lower-density segment acts like a bottleneck for traveling on the route. Peters (2006) used the same functional form presumably out of the same consideration. However, it is possible to be more flexible on the relative importance of the lower vs. higher-density segment. For example, one can make use of the function with constant elasticity of substitution (CES):

\[ f_j = \theta_2 \log \left[ \frac{1}{2} F_g(D)^\sigma + \frac{1}{2} F_{g'}(D)^\sigma \right]^{1/\sigma}. \] (2.5)

As parameter \( \sigma \) moves from \(-\infty\) to \(+\infty\), the CES function morphs from \( \min \{F_g, F_{g'}\} \) to \( \max \{F_g, F_{g'}\} \). At \( \sigma \to 0 \), it converges to \( \sqrt{F_g \cdot F_{g'}} \), so (2.5) reduces to (2.4). Another special case is \( \sigma = 1 \), where it becomes \( \frac{1}{2} F_g + \frac{1}{2} F_{g'} \). As we will see later, the estimate of \( \sigma \) is not significantly away from zero, so (2.4) seems a good approximation. Also notice that in general \( \theta_2 \neq \theta_1 \). One probably should expect \( \theta_2 > \theta_1 \) because for a one-stop route, delays occur not only at the two ends but also at the connection.

The modeling approach here lets density \( F_g \) directly enter the utility function. The main mechanism for this utility specification relies on two presumed channels: (i) flight frequencies are adjusted by carriers to match densities and (ii) consumers generally value higher flight frequencies. Given this, there are alternative approaches to model this mechanism. One alternative is to treat the flight frequency as an explicit choice of carriers, together with the price. The actual flight frequency, instead of \( F \), will enter the utility. However, it is unclear how to do so in an empirically tractable way, particularly because the frequency choices will also put capacity constraints on \( F \). These constraints effectively limit the joint demand across various sets of routes given the network structure. So a carrier’s problem becomes a complicated constrained maximization concerning the entire network.\(^6\) A simpler alternative approach is to estimate a reduced-form relation between \( F_g \) and frequency. While this relation can be interesting in its own right, it seems unnecessary for my purpose because what matters is the combined effect of channel (i) and (ii) rather than the separate effect of each. Moreover, there could be other mechanisms through which the density enters utility, and \( f_j \) as defined by (2.3) and (2.4) can capture these mechanisms too.\(^7\)

With these said, from the perspective of providing a foundation for my modeling approach, it is still important to make sure that channel (i) and (ii) both have empirical standings. Fortunately,

\(^6\)Berry and Jia (2010) started their work by taking this approach, but deemed it infeasible in the end.

\(^7\)One such mechanism is the aircraft size. Carriers respond to higher densities not only by increasing frequencies but also, to a lesser extent, by increasing aircraft sizes (Givoni and Rietveld, 2009); larger aircraft are usually more comfortable so it can have a positive effect on demand too.
Table 1: Carrier Choices and Passenger Value of Flight Frequency

<table>
<thead>
<tr>
<th>Carrier response (freq. increase) to density</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Givoni and Rietveld (2009)</td>
<td>0.65% freq. increase per 1% density increase</td>
</tr>
<tr>
<td>Schipper, et al. (2002)</td>
<td>0.75% ———</td>
</tr>
<tr>
<td>Mohring (1976)</td>
<td>0.5% ———</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Passenger utility for frequency</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Berry and Jia (2010)</td>
<td>0.11, coeff. for freq. in logit utility</td>
</tr>
<tr>
<td>Peters (2006)</td>
<td>0.97, coeff. for log(freq.) in logit utility</td>
</tr>
<tr>
<td>Lijesen (2006)</td>
<td>$0.91 for 1% increase in freq.</td>
</tr>
</tbody>
</table>

Notes: The result from Lijesen (2006) is presented at 2 departures per day, roughly the median level of flight frequency in the city-pair markets that I consider.

the literature has separately documented both the channels. Table 1 summarizes the results from various studies. The studies on channel (i) are mainly from transportation research. Civoni and Rietveld (2009) and Schipper et al. (2002) are empirical studies that relate data on actual frequencies and aircraft sizes to traffic densities. Mohring (1976) includes a well-known theoretical study on optimal frequency in transportation economics. Together, they conclude that carriers increase the flight frequency by $0.5 \sim 0.75\%$ for each 1% increase in traffic density (the discrepancy is fulfilled by increases in aircraft size). The studies on channel (ii) include Berry and Jia (2010) and Peters (2006), which are two discrete-choice estimations of air-travel demand that I largely follow in this paper. Their utility specification includes actual flight frequency. Lijesen (2006) is a more micro-level study; in transportation research, the value of frequency is mainly motivated by the so-called schedule delay, which is the gap between the flight time (e.g., 6AM) and the passenger’s desired time of travel (e.g., 10AM). Following this motivation, Lijesen (2006) conducts a conjoint analysis to measure passenger’s preferences for flying at various times of the day, and then calculates the value of frequency in terms of how it reduces schedule delays. He also compares his result with several earlier studies, including Morrison and Winston (1985). Overall, the literature has found that there is a significant willing to pay for higher frequencies.

In addition to these qualitative supports for the two channels, the quantitative results from these studies also turn out to support my own estimate of $\theta$. I explain this in more detail in Section 4.

Back to the specification of (2.2), I follow previous studies to let $\varepsilon_{ij}$ have the distribution necessary to generate the nested-logit probabilities. All the routes in the market are nested against an “outside option” of not flying. Denote the non-idiosyncratic part of $u_{ij}$ by $v_j(p_j, \xi_j, D) \equiv \beta' x_j - \alpha p_j + f_j(D; \theta, \sigma) + \xi_j$. The demand for route $j$ is given by

$$
\Psi_j(p, \xi, D) = M_m \cdot \frac{S^\lambda}{1 + S^\lambda} \cdot \frac{e^{v_j(p_j, \xi_j, D)/\lambda}}{S},
$$

(2.6)
where

\[ S \equiv \sum_{k \in m} e^{v_k(p_k, \xi_k, D)} / \lambda. \]

In the above expressions, \( \lambda \) is the nesting parameter, \( k \in m \) denotes that route \( k \) serves the city-pair market \( m \), and \( M_m \) denotes the market size of \( m \). Following previous studies, \( M_m \) is defined as the geometric mean of the population of the two end-point cities in the market.

To complete the demand model, notice that the demand vector itself, \( D \), is an input of the above expression for demand, \( \Psi(p, \xi, D) \). A natural way of modeling such circular relation is using a fixed point. I say that \( D \) is the demand predicted by the model iff \( D_j = \Psi_j(p, \xi, D) \) and \( D_j > 0 \) for all routes \( j = 1, ..., n \). In other words, \( D \) is a fixed point \( \Psi(p, \xi, \cdot) \). I discuss the uniqueness of the fixed point in the appendix. It is assumed in the rest of the paper that \( \Psi(p, \xi, \cdot) \) has a unique fixed point.

The fixed point essentially requires that the model-predicted demand is consistent with the densities (and implicitly, the frequencies) that are used to produce this prediction. The fixed point can be computed by iterating \( \Psi(p, \xi, \cdot) \). It is instructive to use the iteration process to illustrate the properties of the fixed point. However, it should be noted that the model is not imposing the iterations as the actual process by which carriers adjust their flight frequencies.

Figure 2 illustrates a price drop on US Airways’ non-stop route \( j \) between Atlanta and Chicago. For illustration purpose, here I ignore the competition between routes. The three plots, from top to bottom, mark out the links in US Airways’ network that will see density increases with one more iteration of \( \Psi \). The impact starts with US Airways’ Atlanta-Chicago link \( g \). The density increase on \( g \) further raises the demand for \( j \), it also raises the demand for all the one-stop routes using \( g \). As a result, the impact spreads to \( g \)’s adjacent links. The impact then spreads farther from these adjacent links to their adjacent links. The iterations will eventually settle down to the fixed point. This process suggests that the total demand increase is larger than what would be predicted by a model that does not account for the density mechanism. Such a model holds \( f \) fixed at its initial level so the impact of a price change would not go beyond the first iteration. In other words, by incorporating the density mechanism, the model implies a larger price elasticity. One caveat, however, is that the competition between routes is ignored. If competition is taken into account, there may be density decreases on some links. I offer more on this point in Section 5.2.

An implicit assumption behind the fixed point is that carriers simultaneously set price and flight frequency (with the latter implicitly accounted for by changes in density). One could, perhaps, argue for a model with two stages where frequencies and prices are set sequentially. First, this would require treating frequency as an explicit choice, which seems intractable as discussed before. Second, more importantly, such two-stage setting is not required by the institutional setup. Airline carriers have short-term flexibilities with respect to both prices and schedules. In fact, for example, Norman and Strandenes (1994): “... there is nothing in the technology or the institutional arrangements that suggests that capacities must be decided prior to pricing decisions – airlines have great flexibility,
Notes: The plots display the sequence of impact originated from a price drop on the direct route between Chicago and Atlanta on US Airways’ network. The impacted links are shown with a darker color. The axes are longitudes and latitudes. For simplicity, competition effects are ignored.

Figure 2: The Spread of the Density Effect
it is common in transportation research to model price and frequency as simultaneous decisions (e.g., Dobson and Lederer, 1993; Adler, 2001). The fixed-point approach allows me to account for changes in flight frequencies without deviating too far from the standard empirical framework for analyzing differentiated goods markets (Berry et al. 1995). In particular, the carrier behaviors can still be modeled with a Nash-Bertrand equilibrium in price alone, which I now turn to.

2.3 Supply

To model carrier’s behavior, one first needs to specify its costs. Let \( z_g \) be a vector of the observed characteristics of link \( g \) (e.g., its length). Let \( C(z_g; \gamma) \) be the marginal cost of operating link \( g \). The marginal cost of route \( j \) is then the sum of the marginal costs of its segments, plus a route-specific unobserved component \( \omega_j \):

\[
mc_j = \sum_{g \in j} C(z_g; \gamma) + \omega_j. \tag{2.7}
\]

As an alternative specification, I allow the density \( F_g \) to enter the marginal cost so that the cost function becomes \( C(z_g, F_g; \gamma) \). The reason for this specification is to account for economies of density (Caves et al. 1984), which suggests that the marginal cost of operating a link decreases with its traffic density. However, note that with the only exception of Berry, Carnell and Spiller (2006), none of the mentioned discrete-choice studies of the industry has included density in their marginal cost specifications.

To complete the model, I assume a Nash-Bertrand equilibrium in price. The equilibrium can be characterized by a set of first-order conditions. Let \( D(p, \xi) \) be the fixed point of \( \Psi(p, \xi, \cdot) \). For each route \( j \) that belongs to some carrier \( c \), denoted as \( j \in c \), we have

\[
D_j(p, \xi) + \sum_{k \in c} (p_k - mc_k) \frac{\partial D_k(p, \xi)}{\partial p_j} = 0. \tag{2.8}
\]

Notice that, unlike in the standard framework for differentiated goods markets, the summation in the above equation is across all the routes belonging to carrier \( c \), not just those in the same city-pair market with \( j \). The reason is exactly what Figure 2 has illustrated: the city-pair markets are all tied together through the network structure; so a price change on a particular route affects the routes in the other markets.

3 Estimation

The estimation largely follows previous discrete-choice applications but with two important differences. First, density is endogenous, which requires a separate instrument aside from the usual instrument for price. Second, demand is modeled as a fixed point, which affects how price elasticities are computed. This subsequently affects the way that the marginal costs are inferred from even in the short term, with respect to prices, capacities, and schedules.”
the pricing behaviors of the carriers. Below, I give the details of the estimation procedure. The appendix further offers some results from the Monte Carlo experiment to verify that the estimator is well-behaved. It also gives details on how the standard errors are bootstrapped.

3.1 Estimation Algorithm

In the data, one observes the network structure, price \( p \), and demand \( D \). For the nested logit model, Berry (1994) provides a closed-form expression to back out the non-idiosyncratic part of utility, \( v_j \), from the observed demand. Using that expression, one can re-write the utility specification (2.2) in the following way:

\[
\log \left( \frac{D_j}{M_m - \sum_{k \in m} D_k} \right) = (1 - \lambda) \log \left( \frac{D_j}{\sum_{k \in m} D_k} \right) + \beta' x_j - \alpha p_j + f_j(D; \theta, \sigma) + \xi_j. \tag{3.1}
\]

Estimation of the demand parameters basically boils down to the regression of this equation. On the right side of the equation, there are three terms that are endogenous in the model: \( \log(D_j/\sum_{k \in m} D_k), p_j, \) and \( f_j \). Obtaining unbiased estimates requires instruments for these terms. I give the details about the instruments below in Section 3.2. I carry out the regression with the standard procedure of generalized method of moments. Before moving to the supply side, it is useful to notice that the regression also provides the value for \( \xi \) as the residuals. With this \( \xi \) and the observed \( p \), the observed demand vector is by construction a fixed point of \( \Psi(p, \xi, \cdot) \).

Now moving to the supply side estimation, the first step is to infer the marginal costs from carriers’ pricing behaviors (or the Nash-Bertrand assumption). The pricing first-order condition (2.8) in matrix form is

\[
D + \Delta(p - mc) = 0,
\]

where matrix \( \Delta \) is defined in the following way:

\[
\Delta_{jk} = \begin{cases} 
\frac{\partial D_k(p, \xi)}{\partial p_j} & \text{if } k, j \in c \text{ for some } c, \\
0 & \text{otherwise.}
\end{cases} \tag{3.2}
\]

Recall that the demand function \( D(p, \xi) \) above is implicitly defined as the fixed point of \( \Psi(p, \xi, \cdot). \) To numerically compute \( \partial D_k(p, \xi)/\partial p_j \), it is most straightforward to use the finite difference formula, which boils down to evaluating \( D(p', \xi) \) at a perturbed price \( p' \) that is different from the observed \( p \). This can be done by iterating \( \Psi(p', \xi, \cdot) \) until it converges.

There is, however, an alternative method to differentiate the demand function. It can be faster in some situations because it avoids the fixed-point iterations. Notice that the implicit function theorem says that \( D(p, \xi) \) is locally unique in \( p \) and

\[
\frac{\partial D(p, \xi)}{\partial p} = \left[ I - \frac{\partial \Psi(p, \xi, D)}{\partial D} \right]^{-1} \frac{\partial \Psi(p, \xi, D)}{\partial p},
\]
provided that the usual invertibility condition for the implicit function theorem is satisfied. Both of the derivatives \( \partial \Psi / \partial D \) and \( \partial \Psi / \partial p \) on the right side can be computed by finite difference formulas without any fixed-point iteration. However, this method does require the inversion of \((I - \partial \Psi / \partial D)\), which is a matrix of size \(n\) (i.e., the number of routes across all carriers). The inversion becomes very difficult computationally for larger \(n\), so this alternative method typically works better for smaller-size networks.

Once \(\Delta\) is computed, the marginal costs can be backed out as

\[ mc = p + \Delta^{-1}D. \]

Although the size of \(\Delta\) is also \(n\), it can be arranged into a block diagonal matrix with each block corresponding to one carrier. Then the computation of \(\Delta^{-1}D\) can be carried out block by block. With the value of \(mc\) backed out, the supply side estimation boils down to a regression of the marginal cost equation (2.7).

Finally, it is worthwhile to point out how the estimation would work if the density mechanism is ignored. As discussed in Section 2.2, this means that one treats \(f_j\) as a fixed variable rather than a function of \(D\). Accordingly, the demand is no longer a fixed point and \(\Delta\) will be defined as:

\[
\Delta_{jk} = \begin{cases} 
\frac{\partial \Psi_k(p, \xi, D)}{\partial p_j} & \text{if } k, j \in c \text{ for some } c, \\
0 & \text{otherwise.}
\end{cases}
\]

This leads to the way that the standard discrete-choice framework backs out the marginal costs. Later, I will compare the estimates from the above \(\Delta\) and the estimates from the \(\Delta\) in (3.2).

### 3.2 Instruments and identification

What can be used as instruments hinges on what are exogenous in the model. Since the network structure is treated as an exogenous primitive of the model, I derive all the instruments from it. The exogeneity of the network structure is also assumed in most previous discrete choice studies of the airline industry (e.g., Berry, Carnell and Spiller, 2006; Peters, 2006; Berry and Jia, 2010; Aguirregabiria and Ho, 2012). However, in a “larger” model where airline entry decisions are endogenous of subsequent demand and cost-side shocks, the network structure is no longer a valid source of instruments and the identification needs to exploit other primitives (see, e.g., Ciliberto and Williams, 2014).

Specifically, at least one instrument is needed for each of the three endogenous terms in (3.1), and they are constructed as follows. First, for the term in front of \(\lambda\), I use the number of the routes in a market as the instrument. Intuitively, \(\lambda\) is identified from the change in the share of the outside good as the number of routes in a market varies across markets. The smaller \(\lambda\) is, the less elastic the market share of the outside good is with respect to the number of inside goods.

Second, for the price, I use two standard instruments: (i) the number of carriers in the same
market with \( j \) and (ii) the number of rivals’ routes in the same market with \( j \). This follows the idea of Berry et al. (1995) to exploit variables measuring the degree of competition at the market level. Intuitively, the more competitive a market is, the lower the prices will be in that market.

Third, for the density term, I exploit the network “centrality” of a link. Specifically, for any link \( g \) belonging to a certain carrier, let \( E_g \) be the number of this carrier’s links that attach to either of the two end cities of \( g \). To provide an example, Figure 3 reproduces the carrier 1’s network in Figure 1. For the link between city 1 and 5, there are 4 links connected to either of the end cities. As a comparison, for the link between city 1 and 2, there are only 7 such links. With everything else being equal, one should expect a higher density on the link between city 1 and 2 than on the link between city 1 and 5. Essentially, a link with a higher \( E_g \) is more utilized by one-stop routes and thus more likely to bear a larger density of passengers. The instruments for \( f_j \) can be easily constructed from \( E_g \) as \( \log(\sqrt{E_gE_g'}) \), \( \log(E_g + E_g') \), and \( \log(E_g^{-1} + E_g'^{-1}) \), which correspond to the CES function in (2.5) at \( \sigma = 0, 1, \) and \(-1\), respectively.

Finally, it is important to note that the identification strategy requires some overlap between routes, otherwise I would have a “reflection problem.” To illustrate, consider a purely point-to-point network (i.e., a carrier attaches at most one link to any city). Each link \( g \) is used by exactly one route \( j \) and \( D_j = F_g \). As a result, the demand estimation essentially regresses the observed demand on itself. This reflection problem is similar to the one described in Manski (1993) in the identification of peer effects in social groups. Fortunately, the hub-spoke structure provides plenty of asymmetry between demand and density. In this regard, the demand estimation in this paper breaks the reflection problem in a similar way as in Bramoullé et al. (2009), which makes use of the asymmetries in social networks to identify peer effects.
4 Data and Results

4.1 Data

The Airline Origin and Destination Survey (DB1B) is a 10% sample of all the airline tickets in the U.S., collected by the Bureau of Transportation Statistics. This paper uses the DB1B-coupon and the DB1B-market data for the last quarter of year 2012 and the first quarter of year 2013. During this period, some carriers were only contracting for other carriers, i.e., they did not sell tickets directly to passengers. The flights of these carriers are incorporated into the carriers for which they operate.\textsuperscript{9}

The analysis in this paper restricts the attention to the 12 largest carriers and their networks between the 100 most visited cities in the U.S. This covers over 87% of the passenger-trips observed in the DB1B data. I regard a link as operational iff at least 360 passengers appear in the DB1B data. Given that the data is a 10% sample and covers a period of two quarters, this cutoff roughly corresponds to the number of passengers that would be carried on a weekly flight by a medium-size jet. Price $p_j$ is the average fare observed on route $j$. I eliminate tickets with unusually low or high fares as they may represent coding errors. A route is included in the estimation only if its length is not longer than 2.5 times the point-to-point distance and the data records at least one passenger traveling on it.\textsuperscript{10} Intra-Hawaii routes are excluded. In the end, my sample includes 44,121 routes and accounts for about 81 percent of the passenger trips in the DB1B data.

Table 2 displays some summary statistics for the five major carriers. It is interesting to note that Southwest Airlines, while having the largest number of links, serves the fewest markets. This is because Southwest’s network exhibits more of a point-to-point rather than hub-spoke structure when compared with the other major carriers. It is well known in the industry that Southwest has a focus on direct services.

I make the following empirical specifications for the vector of route characteristics, $x_j$, and the vector of link characteristics, $z_g$. Vector $x_j$ includes an intercept, the carrier dummies, point-to-point distance between the two end cities, square of that distance, a dummy indicating whether either city in the market is a tourist destination\textsuperscript{11}, a dummy for one-stop route, the difference between the route’s length and the point-to-point distance, and finally, in the case that the operator is a legacy carrier, a dummy indicating whether either city in the market is a hub of the carrier.\textsuperscript{12}

\textsuperscript{9}I acknowledge that some carriers also operate internationally, and these operations bring passenger flows onto the U.S. networks. However, data on international itineraries is not publicly available.

\textsuperscript{10}The framework of Berry et al. (1995) does not allow a product with zero market share. Here, I follow the practice in previous studies and exclude the routes with no observed passengers. Gandhi et al. (2015) suggest a method to deal with the zero market share problem. However, it is not straightforward to apply their method because there is the additional issue that $p_j$ is not defined on a route with no observed passengers.

\textsuperscript{11}Following previous literature, tourist destinations include Atlantic City, Charlotte Amalie, Las Vegas, New Orleans, and any city in Florida and Hawaii.

\textsuperscript{12}The populations in the two end cities have already entered the model as market sizes $M_m$; thus are not included in $x_j$. The literature typically does not include in $x_j$ the GDP per capita at the end cities. I estimated a version of the model which adds the GDP per capita in $x_j$, the associated coefficient is statistically significant but has very little economic significance; the estimates of the other parameters in the model remain almost unchanged.
Table 2: Summary Statistics for the Five Major Carriers

<table>
<thead>
<tr>
<th>Carrier</th>
<th>Links</th>
<th>Routes</th>
<th>Markets served</th>
<th>Total flow (million)</th>
<th>Total demand (million)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Southwest</td>
<td>573</td>
<td>9,070</td>
<td>1,721</td>
<td>51.8</td>
<td>42.6</td>
</tr>
<tr>
<td>Delta</td>
<td>422</td>
<td>10,097</td>
<td>3,728</td>
<td>43.6</td>
<td>31.9</td>
</tr>
<tr>
<td>United</td>
<td>462</td>
<td>11,076</td>
<td>3,877</td>
<td>31.2</td>
<td>24.2</td>
</tr>
<tr>
<td>American</td>
<td>237</td>
<td>5,074</td>
<td>2,967</td>
<td>26.4</td>
<td>20.2</td>
</tr>
<tr>
<td>US Airways</td>
<td>258</td>
<td>5,233</td>
<td>2,803</td>
<td>27.8</td>
<td>19.4</td>
</tr>
</tbody>
</table>

Notes: Compiled from the DB1B data for 2012Q4 and 2013Q1. The networks are restricted to the top 100 cities. The total flow for a carrier is the sum of the densities across all of its links.

Table 3: Summary Statistics

<table>
<thead>
<tr>
<th>Route level:</th>
<th>Mean</th>
<th>SD</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point-to-point distance (1,000 miles)</td>
<td>1.37</td>
<td>0.84</td>
<td>0.06</td>
<td>5.97</td>
</tr>
<tr>
<td>Tourism dummy</td>
<td>0.29</td>
<td>0.45</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Hub dummy</td>
<td>0.17</td>
<td>0.38</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>one-stop dummy</td>
<td>0.94</td>
<td>0.23</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Extra length (1,000 miles)</td>
<td>0.35</td>
<td>0.36</td>
<td>0</td>
<td>3.44</td>
</tr>
<tr>
<td>Price, $p_j$ ($100)</td>
<td>2.62</td>
<td>1.02</td>
<td>0.03</td>
<td>9.96</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Link level:</th>
<th>Mean</th>
<th>SD</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point-to-point distance (1,000 miles)</td>
<td>0.98</td>
<td>0.69</td>
<td>0.06</td>
<td>4.96</td>
</tr>
<tr>
<td>Density, $F_g$ (100,000 passengers)</td>
<td>0.92</td>
<td>1.26</td>
<td>0.003</td>
<td>25.9</td>
</tr>
</tbody>
</table>

Notes: There are 44,121 routes and 2,419 links. The point-to-point distance is the great-circle distance between the two end cities. The extra length is the difference between the travel length and point-to-point distance. For a non-stop route, the extra length is zero.
Vector $z_g$ includes the carrier dummies and point-to-point distance between the two end cities. These specifications have been common in previous discrete-choice studies. Table 3 provides the summary statistics.

### 4.2 Parameter estimates

#### Demand side

Table 4 displays the estimates for the demand-side parameters. The first column presents the benchmark estimates. All the estimates have expected signs. In particular, the effect of the point-to-point distance is estimated to be inverted U-shaped. This is intuitive: at very short distances, air travel has to compete with cars and trains, so demand initially grows with distance. As distance increases further, travel becomes less pleasant and demand starts to decrease.

The price coefficient is estimated to be 0.65. The implied aggregate price elasticity, which is the percentage change in total demand when all products’ prices increase by 1%, is 1.19 without taking into account the density mechanism (i.e., $f_j$ is fixed at the level in the data). The coefficients in front of the density terms, $\theta_1$ and $\theta_2$, are both estimated to be significantly positive. To get a sense of their economic significance, note that the aggregate price elasticity goes up to 1.89 under the full model. This larger elasticity conforms with the previously discussed intuition for the fixed point (see Section 2.2). Moreover, one can compare these model-implied elasticities to Gillen et. al. (2003), which conducts a survey of cross-sectional studies on the elasticity of air-travel demand. Their estimates ranged from 0.181 to 2.01, with a median of 1.33. So my results seem reasonable.

Also notice that the demand effect of density is larger on one-stop routes than that on non-stop routes (i.e., $\theta_2 > \theta_1$). This is expected. On non-stop routes, the preference for a higher flight frequency mainly comes from two reasons. First, it reduces the gap between the flight’s scheduled time and the passenger’s desired time of travel (i.e., the schedule delays). Second, a higher frequency makes it easier for a passenger to reschedule a missed flight. Both reasons apply to one-stop routes too. However, the second reason seems more pronounced when there is a connecting flight to catch. In addition, higher segment frequencies tend to reduce the transferring time for one-stop routes.

To get an even better sense of whether the estimates of $\theta$ are plausible, we can compare them with the numbers in Table 1, which summarizes the results on the two channels underlying the density mechanism. Peters (2006) estimates a coefficient of 0.97 in front of log flight frequency. Given that carriers typically increase frequency by $0.5 \sim 0.75\%$ for each $1\%$ increase in density, this coefficient for log frequency translates into a coefficient of $0.48 \sim 0.72$ for log density. The result by Lijesen (2006), on the other hand, indicates a coefficient of at $0.29 \sim 0.44$ for log density. This somewhat smaller coefficient is not surprising, given that the value of frequency in Lijesen (2006) is measured solely from the reduction in schedule delays (not considering the convenience

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13Given the coefficient estimate of price in Table 4, one utile equals $100/0.65 = 154$. So a willingness to pay of $80.91 (= 0.0059$ utile) for $1\%$ increase in frequency translates into a coefficient of 0.59 in front of log frequency. However, a $1\%$ increase in density only leads to $0.5 \sim 0.75\%$ increase in frequency. So for log density, the coefficient should be 0.59 multiplied by $0.5 \sim 0.75$. 

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<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>No IV</th>
<th>CES</th>
<th>Non-Tourism</th>
<th>Non-Congested</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nesting parameter ($\lambda$)</td>
<td>0.56 (.01)</td>
<td>0.55 (.01)</td>
<td>0.57 (.01)</td>
<td>0.56 (.004)</td>
<td>0.56 (.004)</td>
</tr>
<tr>
<td>Intercept</td>
<td>-2.83 (.09)</td>
<td>-3.33 (.03)</td>
<td>-2.82 (.10)</td>
<td>-2.97 (.06)</td>
<td>-2.74 (.06)</td>
</tr>
<tr>
<td>Carrier dummies</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Point-to-point distance</td>
<td>0.28 (.02)</td>
<td>0.28 (.02)</td>
<td>0.28 (.03)</td>
<td>0.56 (.03)</td>
<td>0.24 (.02)</td>
</tr>
<tr>
<td>Point-to-point distance $^2$</td>
<td>-0.031 (.006)</td>
<td>-0.074 (.006)</td>
<td>-0.032 (.007)</td>
<td>-0.14 (.01)</td>
<td>-0.020 (.006)</td>
</tr>
<tr>
<td>Tourism dummy</td>
<td>0.32 (.01)</td>
<td>0.37 (.01)</td>
<td>0.32 (.01)</td>
<td>-</td>
<td>0.30 (.01)</td>
</tr>
<tr>
<td>Hub dummy</td>
<td>0.07 (.02)</td>
<td>-0.17 (.02)</td>
<td>0.05 (.02)</td>
<td>0.06 (.02)</td>
<td>0.13 (.02)</td>
</tr>
<tr>
<td>One-stop dummy</td>
<td>-1.98 (.03)</td>
<td>-2.17 (.02)</td>
<td>-1.97 (.03)</td>
<td>-1.93 (.04)</td>
<td>-1.90 (.04)</td>
</tr>
<tr>
<td>Extra route length</td>
<td>-1.09 (.04)</td>
<td>-1.29 (.02)</td>
<td>-1.08 (.04)</td>
<td>-1.10 (.03)</td>
<td>-1.04 (.03)</td>
</tr>
<tr>
<td>Price ($\alpha$)</td>
<td>0.65 (.05)</td>
<td>0.19 (.01)</td>
<td>0.64 (.05)</td>
<td>0.64 (.02)</td>
<td>0.70 (.03)</td>
</tr>
<tr>
<td>Non-stop density ($\theta_1$)</td>
<td>0.33 (.02)</td>
<td>0.60 (.01)</td>
<td>0.34 (.02)</td>
<td>0.37 (.04)</td>
<td>0.30 (.05)</td>
</tr>
<tr>
<td>One-stop densities ($\theta_2$)</td>
<td>0.49 (.01)</td>
<td>0.75 (.01)</td>
<td>0.50 (.01)</td>
<td>0.53 (.01)</td>
<td>0.45 (.01)</td>
</tr>
<tr>
<td>Substitution ($\sigma$)</td>
<td>-</td>
<td>-</td>
<td>0.10 (.27)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Observations</td>
<td>44,121</td>
<td>44,121</td>
<td>44,121</td>
<td>31,391</td>
<td>34,603</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.82</td>
<td>0.87</td>
<td>0.82</td>
<td>0.85</td>
<td>0.82</td>
</tr>
<tr>
<td>Cragg-Donald stat.</td>
<td>479.3</td>
<td>-</td>
<td>-</td>
<td>412.6</td>
<td>346.4</td>
</tr>
</tbody>
</table>

Notes: The first column is the benchmark case. The 2nd column differs by not instrumenting for price or density. The 3rd column differs by using the more flexible CES specification for $f_j$. The 4th column runs the regression on the sub-sample of routes that do not involve tourist destinations. The 5th column regresses only the sub-sample of routes that do not involve destinations with congested airports (defined as FAA slot-controlled airports).

The standard errors for Columns 1-3 are bootstrapped (see Appendix). The standard errors for Columns 4-5 are asymptotic. The $R^2$ is defined on (3.1) as 1 minus the ratio between the residual variance and total variance.
of rescheduling or shorter wait time for transferring flights). Nevertheless, overall, these back-of-the-envelope calculations suggest that my estimate of $\theta$ is in line with the existing results in the literature.

The second last row in Table 4 reports the Cragg-Donald test for weak instruments. It is a generalization of the first stage $F$-statistic to the case of more than one endogenous regressor (see Stock, Wright and Yogo 2002). The statistic clearly indicates that the instruments are not weak.

Column 2 of Table 4 displays the estimates when neither price nor density is instrumented. Notice that in this case the price coefficient $\alpha$ is under-estimated. This bias direction is expected because it is optimal for carriers to price up a route $j$ if $\xi_j$ is higher, resulting in a positive correlation between $p_j$ and $\xi_j$. Also notice that $\theta_1$ and $\theta_2$ are over-estimated. This is expected because $f_j$ increases with $D_j$, which increases with $\xi_j$. This results in a positive correlation between $\xi_j$ and $f_j$.

Column 3 explores the specification of $f_j$ given by (2.5), which uses a CES function to weight the segment densities on a one-stop route in a more flexible way. The substitution parameter $\sigma$ is estimated to be statistically insignificant. This could be because the instruments are not strong enough to identify $\sigma$. Unfortunately, to my knowledge, there is not yet an established test for weak instruments for non-linear case (parameter $\sigma$ enters the estimation non-linearly). Nevertheless, with the point estimate of $\sigma$ close to 0, the geometric mean specification used in the benchmark case should be a good approximation.

In general, the density effect can be heterogeneous across markets and routes. Column 4-5 explore this by focusing on two sub-samples of routes. Column 4 runs the regression on the routes that do not involve tourist destinations. The estimates of both $\theta_1$ and $\theta_2$ are larger compared with the benchmark case. This is intuitive as tourists are less sensitive to schedule delays than business travelers. Column 5 focuses on the routes that do not pass congested airports. The estimates of both $\theta_1$ and $\theta_2$ are smaller compared with the benchmark case. This is again intuitive because it is more likely to experience delays or cancellations at congested airports, in which case having a next flight soon will be very valuable. Generally, these types of heterogeneity can be handled with random coefficients and interaction terms. However, given the complexity of the computations in the counterfactuals, I choose to stay with the benchmark model, which captures the average degree of the density effect.

Finally, Table 6 displays the coefficient estimates for the dummies of the five major carriers. With everything else equal, passengers prefer flying with the legacy carriers to flying with Southwest, which is expected given that Southwest is a low-cost carrier.

**Supply side** Table 5 displays the results of the supply side estimation. The estimates correspond to parameter $\gamma$ for the cost function $C(\cdot)$ (see Section 2.3). In particular, notice that these are marginal cost estimates at the link or segment level rather than the route level. The first column displays the benchmark case; the marginal cost of operating a link starts with a base of 75 dollars and then increments by 4.5 cents per mile. This implies that the marginal cost of the direct service
### Table 5: Cost Side Parameter Estimates

<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>Fixed $f_j$</th>
<th>Density</th>
<th>Density IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>0.75 (.02)</td>
<td>0.27 (.05)</td>
<td>0.83 (.03)</td>
<td>0.69 (.12)</td>
</tr>
<tr>
<td>Carrier dummies</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Point-to-point distance</td>
<td>0.45 (.01)</td>
<td>0.43 (.01)</td>
<td>0.45 (.01)</td>
<td>0.44 (.04)</td>
</tr>
<tr>
<td>Density</td>
<td>-</td>
<td>-</td>
<td>-0.10 (.02)</td>
<td>-0.050 (.28)</td>
</tr>
<tr>
<td>Density$^2$</td>
<td>-</td>
<td>-</td>
<td>0.014 (.005)</td>
<td>0.013 (.13)</td>
</tr>
<tr>
<td>Density$^3$</td>
<td>-</td>
<td>-</td>
<td>-0.0004 (.0003)</td>
<td>0.00059 (.011)</td>
</tr>
<tr>
<td>Observations</td>
<td>44,121</td>
<td>44,121</td>
<td>44,121</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.300</td>
<td>0.264</td>
<td>0.305</td>
<td>-</td>
</tr>
<tr>
<td>Cragg-Donald stat.</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Notes: For all columns, the dependent variable is the marginal costs backed out under the parameter estimates in Column 1 of Table 4. Column 1 shows the benchmark case. The other columns are different from Column 1 in following ways. Column 2 ignores the density mechanism when backing out the marginal costs. Column 3 adds a polynomial of density but does not instrument the added terms. Column 4 adds a polynomial of density and instruments the added terms with $E_g$, $E_g^2$, and $E_g^3$. All standard errors are bootstrapped (see Appendix).

### Table 6: Dummy Estimates for Major Carriers

<table>
<thead>
<tr>
<th></th>
<th>Demand Side</th>
<th>Cost Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>Southwest</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Delta</td>
<td>0.20 (.02)</td>
<td>0.13 (.02)</td>
</tr>
<tr>
<td>United</td>
<td>0.32 (.03)</td>
<td>0.31 (.02)</td>
</tr>
<tr>
<td>American Airlines</td>
<td>0.14 (.02)</td>
<td>0.11 (.02)</td>
</tr>
<tr>
<td>US Airways</td>
<td>0.42 (.03)</td>
<td>0.24 (.02)</td>
</tr>
</tbody>
</table>

Notes: The coefficients for the Southwest dummy of are set to zero. These estimates are based on the benchmark model. All standard errors are bootstrapped (see Appendix).
between Los Angeles and Philadelphia is about 183 dollars, for example.

As a comparison to the benchmark case, Column 2 displays the estimates if the model ignores the density mechanism. Again, this means that one treats $f_j$ as fixed at the level in the data (see the end of Section 3.1 for technical details). The resulting marginal costs are smaller, with a base of only 27 dollars and an increment of 4.3 cents per mile. Intuitively, this is because the marginal costs are backed out from the firms’ pricing behaviors: As shown above, ignoring the density mechanism tends to lower the model-implied price elasticity. With lower price elasticities, optimal pricing indicates large markups, thus smaller marginal costs.

As discussed by Einav and Levin (2010), one way to decide whether the estimates from a certain model are more reasonable than those from another is to compare them with the accounting numbers. While it is acknowledged that accounting practices generally are not geared toward reporting the economic notion of costs, accounting reports still can be and have been used to complement econometric analysis (e.g., Nevo, 2001).

Because my data is U.S. domestic, here I focus on Southwest, whose operations were virtually domestic only. According to the filing of Southwest for 2013Q1, it flew 23.7 billion passenger miles and incurred $4 billion operation costs, of which $2.1 billion are attributed to fuel, maintenance, aircraft rentals and landing fees, and $1.9 billion are attributed to salary, depreciation and “other operation expenses.” This implies a cost per passenger mile (CPM) of 8.9 ∼ 16.9 cents. The CPM based on my benchmark model is 12.7 cents, which falls well within this range. On the other hand, the CPM based on the model in Column 2 of Table 5 is only 5.8 cents. Berry and Jia (2010) had a similar estimate of 6 cents. More generally speaking, the airline industry is widely known to be unprofitable (Borenstein, 2011). Taking the density mechanism into consideration seems to align the model closer to this fact.

It is worthwhile to mention that for about 31% of the routes in the data, the prices are lower than the estimated marginal costs in the benchmark model. At first glance, this seems erroneous because $p_j < mc_j$ means that the carrier is not breaking even on $j$. However, one should realize that, in a network, a route is no longer an independent product. Lowering the price on one route can benefit the other routes in the network, as explained by Figure 2. Another way to look at it is that carriers have an incentive to keep flight frequencies high to compete for demand, so at times they want to fill seats at an unprofitably low price rather than leaving them empty. This is particularly true for connecting routes, where the marginal cost is high, consumer willingness to pay is low, and each passenger can fill the seats on multiple segments. Indeed, those routes in the data with negative markups are all one-stop routes. Interestingly, Delta once revealed that “connecting traffic is the least profitable for the airline.”

Columns 3 and 4 in Table 5 explore economies of density by adding a polynomial of density to the benchmark model. Polynomial specifications were proposed by Berry et al. (2006).

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Column 3 displays the OLS estimates, while Column 4 instruments the density terms. We see that there is an over-estimate of economies of density by OLS. This is intuitive because a higher $\omega_j$ drives up $p_j$, resulting in a negative correlation between $\omega_j$ and the segment densities of $j$. The OLS estimates for the first two coefficients of the polynomial are statistically significant. The shape of the polynomial is similar to what Berry et al. (2006) estimated for medium-distance flights. However, the improvement of the $R^2$ is small, from 0.3 to only 0.305. The estimates in Column 4 are far from significant. The Cragg-Donald statistic indicates that the instruments are not strong enough to separate the coefficients of the highly correlated terms in the polynomial.\footnote{Berry et al. (2006) did not have this weak instrument problem because they treated flight frequencies as exogenous and used them as instruments for the density terms.} In regards of this, I will fall back to the benchmark model when presenting the counterfactual results. Incorporating economies of density at the point estimates in Column 4 does not qualitatively change the results in my counterfactuals.

Finally, Column 2 of Table 6 displays the coefficient estimates of the cost dummies for the five major carriers. It is seen that the marginal costs of the legacy carriers are generally higher than that of Southwest, which is expected given that Southwest is a low-cost carrier.

5 Value of links

In this section, I use the estimated model to calculate the value of each link in a carrier’s network. I pay special attention to the part of the value that is attributed to the density effect and compare it to the part attributed to the hubbing effect.

5.1 The counterfactual

The calculation of a link’s value amounts to comparing two equilibria: one in a counterfactual where this single link is removed from the carrier’s observed network, and the other where this link is added back to the network. Given how I have estimated the model, the latter of the two equilibria is exactly as observed in the data. The former equilibrium, however, needs to be computed. The computation here is more demanding than in standard discrete-choice models: because the pair-city markets are tied together through the endogenous densities, tens of thousands of prices must be solved simultaneously rather than market by market.\footnote{Actually, such a counterfactual exercise was thought infeasible and thus avoided in Berry and Jia (2010).} I give more computational details in the appendix. While the actual counterfactual is removing a link, as we will see, it is easier to interpret this comparison exercise as a scenario where a carrier adds the link to an existing network. In this case, “the existing network” is the observed network with the link in question removed.

I carry out the comparison exercise for every link of every major carrier. Before I present the summary results across these links, it is instructive to use a particular example to get a sense of how adding a link affects different parts of the network differently. Figure 4 presents the example with the link between Houston and Chicago in United Airlines’ network. When the carrier adds
The plots indicate the links with substantial density changes on United’s network when it adds back the Houston-Chicago link. The upper-left (and lower-left, respectively) plot marks the links on which the density increases (decreases) by more than 250 passengers. The upper-right (and lower-right, respectively) plot marks the links on which the density increases (decreases) by more than 1,000 passengers.

Figure 4: An Example of Counterfactual

(back) this link, it creates a new non-stop route, many one-stop routes, and more importantly, changes the densities throughout its existing network. The two upper plots mark out the United’s links that see a substantial increase in density (>250 passengers for the left plot and >1,000 for the right plot), while the bottom plots mark out the United’s links that see an equally substantial decrease in density. There are several observations. First, most of the links connected to these two cities experience a sizable increase in density. This is intuitive, as these links form connecting routes with the Houston-Chicago link. In addition, the density increases on these links reinforce each other because they share connecting routes among themselves.

Second, the strengthening of the above links leads to the weakening of some other links in United’s network, essentially due to self-competition. One example is the link between Houston and Washington D.C (marked out in the lower right plot). It sees a decrease in density for several reasons. First, it can hardly form a one-stop route with the new link (the Chicago-Houston-D.C. route would be too much of a detour). Second, the Houston-D.C. non-stop route must now compete with the new Houston-Chicago-D.C. route. Third, the Houston-D.C.-Chicago route must now compete with the new Houston-Chicago route. Further, the density decrease on the Houston-D.C. link has a negative effect on the demand for one-stop routes such as Houston-D.C.-Buffalo,
Table 7: Value of Links for Major Carriers

<table>
<thead>
<tr>
<th>Number of links</th>
<th>SW</th>
<th>DL</th>
<th>UA</th>
<th>AA</th>
<th>US</th>
</tr>
</thead>
<tbody>
<tr>
<td>573</td>
<td>422</td>
<td>462</td>
<td>237</td>
<td>258</td>
<td></td>
</tr>
</tbody>
</table>

Total demand change (1,000 passengers)

<table>
<thead>
<tr>
<th></th>
<th>SW</th>
<th>DL</th>
<th>UA</th>
<th>AA</th>
<th>US</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carrier’s new non-stop route</td>
<td>58.4</td>
<td>48.18</td>
<td>37.2</td>
<td>59.4</td>
<td>42.0</td>
</tr>
<tr>
<td>Carrier’s new one-stop routes</td>
<td>32.0</td>
<td>55.2</td>
<td>30.4</td>
<td>52.1</td>
<td>65.9</td>
</tr>
<tr>
<td>Carrier’s existing non-stop routes</td>
<td>6.97</td>
<td>12.2</td>
<td>11.2</td>
<td>17.3</td>
<td>14.4</td>
</tr>
<tr>
<td>Carrier’s existing one-stop routes</td>
<td>0.74</td>
<td>16.9</td>
<td>7.8</td>
<td>18.4</td>
<td>34.4</td>
</tr>
<tr>
<td>Other carrier’s routes in market of entry</td>
<td>-20.9</td>
<td>-16.6</td>
<td>-15.6</td>
<td>-24.6</td>
<td>-12.7</td>
</tr>
<tr>
<td>Other carrier’s routes in other markets</td>
<td>-28.5</td>
<td>-47.5</td>
<td>-30.0</td>
<td>-51.0</td>
<td>-58.5</td>
</tr>
</tbody>
</table>

Total profit change ($ million)

<table>
<thead>
<tr>
<th></th>
<th>SW</th>
<th>DL</th>
<th>UA</th>
<th>AA</th>
<th>US</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carrier’s new non-stop route</td>
<td>3.96</td>
<td>3.16</td>
<td>1.93</td>
<td>3.33</td>
<td>2.75</td>
</tr>
<tr>
<td>Carrier’s new one-stop routes</td>
<td>0.21</td>
<td>0.68</td>
<td>0.19</td>
<td>0.45</td>
<td>0.65</td>
</tr>
<tr>
<td>Carrier’s existing non-stop routes</td>
<td>0.74</td>
<td>1.26</td>
<td>0.80</td>
<td>1.46</td>
<td>1.41</td>
</tr>
<tr>
<td>Carrier’s existing one-stop routes</td>
<td>0.28</td>
<td>0.67</td>
<td>0.27</td>
<td>0.54</td>
<td>0.84</td>
</tr>
<tr>
<td>Other carrier’s routes in market of entry</td>
<td>-1.77</td>
<td>-1.23</td>
<td>-1.28</td>
<td>-1.83</td>
<td>-1.21</td>
</tr>
<tr>
<td>Other carrier’s routes in other markets</td>
<td>-1.33</td>
<td>-2.63</td>
<td>-1.69</td>
<td>-2.81</td>
<td>-4.13</td>
</tr>
</tbody>
</table>

Notes: The numbers are averaged across all the links in the corresponding carrier’s network.

which then decreases the density on the D.C.-Buffalo link (marked out in the lower left plot).

Third, the overall effect of the added link on the existing network is positive. While some routes do suffer from self-competition, this is mostly a second-order effect compared with the benefits from the generally higher densities throughout the network. This is seen by comparing the upper and lower plots in Figure 4. Among the links on which density changes by more than 250 passengers, a greater number of links see an increase instead of a decrease. Among the links on which density changes by more than 1,000 passengers, many see an increase while only a handful see a decrease.

5.2 Results

Table 7 summarizes the results for the major carriers. For each carrier, the reported numbers are the averages across all the links of that carrier. The upper part of the table displays about the change in a carrier’s total demand thanks to the addition of a link, and it is broken down into four parts: (i) the demand on the new non-stop route, (ii) the total demand on the new one-stop routes, (iii) the total demand change on the carrier’s existing non-stop routes, and (iv) the total demand change on the carrier’s existing one-stop route. The table also displays the impacts on the rival carriers, which I will discuss later. The lower part of the table is organized in the same way, except it displays about the change in profits instead of demand.

For example, there are 573 links in Southwest’s network. When Southwest adds one of these
links to the rest of them, the profit from the created non-stop route is $3.96 million and the collective profit from the created one-stop routes is $0.21 million, on average. Southwest’s total profit on the existing routes increases by $0.74 (non-stop routes) plus $0.28 (one-stop routes) totaling $1.02 million.

More generally, one can attribute these numbers in Table 7 to the various components that make up a link’s value. The profit from the new non-stop route is the “stand-alone value” of the link, because it would exist even without the rest of the network. The profits from the new one-stop routes constitute one part of the link’s “network value.” This part corresponds to the hubbing effect. The profit increase on the existing routes (both non-stop and one-stop) constitutes the other part of the link’s network value. This part arises because of the density effect. The sum of the stand-alone and network values gives the total increase in profits for the carrier when a link is added to its network, accounting for how all the carriers will adjust both their prices and flight frequencies.

The main result here is that the density effect is much larger than the hubbing effect (e.g., $1.02 vs. 0.21 million for Southwest). In other words, while it is true that a carrier derives significant network value when establishing a link, most of this network value actually comes from the increased flight frequencies rather than the creation of connecting routes. Hence, ignoring the density effect will greatly underestimate a link’s network value. There is one caveat to this conclusion. We have focused on the value (i.e., profits) to the carrier. But if the two effects are defined in terms of the demand (instead of profits) on the respective sets of routes, then the hubbing effect would be larger, as shown in the Table. Intuitively, this is because the margins on connecting routes are generally very low (see Section 4.2) and the hubbing effect is entirely about the connecting routes.

There is a substantial variation in the density effect across carriers. In particular, one may expect a weaker density effect from Southwest, because it operates a relatively more point-to-point instead of a hub-spoke network. Indeed, in terms of demand, Table 7 shows that Southwest has clearly the smallest increase on existing routes (non-stop and one-stop) among the major carriers. However, in terms of profits, one must factor in that Southwest has the lowest marginal costs. Table 7 shows that Southwest’s profit increase on existing routes is still the smallest, but it comes close to that of United, which is estimated to have the largest marginal cost among the major carriers.

There is also a substantial variation across the links within a carrier. Figure 5 shows the distribution of the density effect across the links of United, in terms of both demand and profit. The distribution is skewed, where the right tail contains mostly the links between the carrier’s hubs. Figure 5 also displays how the density effect of a link (in terms of profits) is positively correlated with two link characteristics: (i) the geometric mean of the populations at the two end cities, and (ii) the centrality of the link in the carrier’s network. The centrality equals the

\[18\] Strictly speaking, the new routes also benefit from the density effect. Hence the profit increase on the existing routes should be seen as a lower bound of the density effect.
Notes: The two histograms display the distribution of the network value due to density effect across the 462 links in the United’s network. For the three scatter plots, each dot corresponds to one of the 462 links. The correlations in the scatter plots are 0.35 (top), 0.47 (middle), and 0.68 (bottom), respectively.

Figure 5: Distribution of the Density Effect across Links
number of United’s links that are connected to either of the two end cities. Together, the two characteristics can explain about 30% of the variation in the density effect.

Similarly, one would also expect the hubbing effect to positively correlate with these two characteristics. Consequently, one should expect a positive correlation between the hubbing effect and density effect. This is shown in the last plot in Figure 5. However, the correlation is far from perfect. Hence, ignoring the density effect will not only under-estimate a link’s network value but also distort the ranking of links in terms of their network values. All the patterns in Figure 5 hold for the other major carriers too.

5.3 Impact on the rivals

The economic literature on the industry has long been interested in the role of entry into city-pair markets (e.g., Morrison and Winston, 1990; Berry, 1992; Hendricks et al., 1997; Goolsbee and Syverson, 2008; Aguirregabiria and Ho, 2012). An “entry” corresponds to adding a link between the city pair. The literature has focused on two aspects: the factors that lead to entry decisions and the competitive impact of entry. When examining the latter, the attention has been (naturally) focused on the particular city-pair market where the entry happens. However, from a network perspective, particularly in the presence of the density effect, an important question is to what extent the impact can spread to the other city-pair markets.

Table 7 breaks down the impact on the rival carriers into two parts. The first part sums over all rivals’ routes within the city-pair market of entry, whereas the second part sums over all rivals’ routes outside the market. For example, when Southwest adds (back) one of its links, the entry causes the other carriers to incur a total profit loss of $1.77 million within that city-pair market and a total profit loss of $1.33 million outside that market. When any legacy carrier enters a market, the rival carriers incur a larger loss outside that market than within that market. In this sense, the competitive impact of a market entry is mostly network-wide rather than local. There is, however, one caveat. There is typically a much greater number of routes outside a market than within the market. Hence, although the aggregate impact is larger outside the market, the impact per route is usually larger within the market. So, whether it is necessary to adopt a network-wide perspective depends on how much one cares about the aggregate impact vs. the impact on individual routes.

6 Concluding remarks

The significance of the density effect has several implications. With respect to explaining the emergence of the hub-spoke structure, it calls for more emphasis on how such a structure provides carriers demand-side benefits by concentrating traffic densities. With respect to airline network design, it indicates that the optimal location for a new link depends on how much it will increase densities instead of just how many connecting routes that it will create. With respect to the exam-
ination of the competitive effects of entry, it suggests taking a more network perspective instead of solely focusing on the city-pair market where the entry occurs. Aside from these substantive implications, the modeling approach of this paper can be of use for future research too. This paper develops a tractable model that accounts for both price competition and frequency adjustments over the entire U.S. network. The model can be used for other counterfactuals where frequency changes potentially play an important role. More generally, it sheds lights on the applications of the discrete-choice framework on other types of networked markets with demand externalities.
7 Appendix

7.1 Uniqueness of the fixed point

Here I consider the mapping $\Psi(p, \xi, \cdot)$ over the space of positive vectors. Below I present a parameter condition $(\theta_1, \theta_2 < \frac{1}{2})$ which guarantees a unique fixed point. The uniqueness requires no restrictions on the network topology (e.g., whether it is a point-to-point or hub-spoke network). The condition is a sufficient one and the proof accounts for the worst case scenarios that could break the convergence to a unique fixed point. The downside of this is that the condition is stringent and estimated parameter values fail to satisfy it. Nevertheless, in my experiences, under the estimated parameter values, iteration of $\Psi(p, \xi, \cdot)$ always converges to the same point no matter how I change the starting point. The main difficulty in extending the proof to include a larger range for $\theta$ is the complexity of competition effects. Suppose that one is willing to assume away the competition between routes, so that each route competes with the outside good separately ($\Psi_j = M_m e^{v_j}/(1 + e^{v_j})$). Then one can relax the condition to $\theta_1, \theta_2 < 1$ by following essentially the same proof.

Proposition 1. If $\max\{\theta_1, \theta_2\} < \frac{1}{2}$, then $\Psi(p, \xi, \cdot)$ has a unique fixed point. Moreover, iteration of $\Psi(p, \xi, \cdot)$ always converges to the fixed point.

Proof. First, let us define a “distance” measure $\Xi$ between any two positive vectors of the same length. Note that $\Xi$ is always greater than or equal to 1, and $\Xi = 1$ implies $V = V^*$.

$$\Xi(V, V^*) \equiv \max \left\{ \frac{V_1}{V_1^*}, \ldots, \frac{V_n}{V_n^*}, \frac{V_1^*}{V_1}, \ldots, \frac{V_n^*}{V_n} \right\}.$$ Let $D$ and $D^*$ be two positive demand vectors. With the specification in (2.1), it is seen that

$$\Xi[F(D), F(D^*)] \leq \Xi[D, D^*].$$

Let $f(D; \theta, \sigma)$ be the vector that collects $f_j$, which is defined as in (2.3) and (2.4). Then,

$$\Xi\left[ e^{f(D; \theta, \sigma)}, e^{f(D^*; \theta, \sigma)} \right] \leq \Xi\left[ F(D), F(D^*) \right]^{\max\{\theta_1, \theta_2\}},$$

where the exponential of a vector refers to the vector collecting the exponential of each element. Let $v(p, \xi, D)$ be the vector that collects $v_j$. Then,

$$\Xi\left[ e^{v(p, \xi, D)}, e^{v(p, \xi, D^*)} \right] = \Xi\left[ e^{f(D; \theta, \sigma)}, e^{f(D^*; \theta, \sigma)} \right].$$

It is not difficult (though not easy either) to see that the nested logit specification (2.6) implies

$$\Xi[\Psi(p, \xi, D), \Psi(p, \xi, D^*)] \leq \Xi\left[ e^{v(p, \xi, D)}, e^{v(p, \xi, D^*)} \right]^{2/\lambda}.$$
Putting everything so far together, we have

\[\Xi \left[ \Psi(p, \xi, D), \Psi(p, \xi, D^*) \right] \leq \Xi(D, D^*)^{2 \max(\theta_1, \theta_2)/\lambda}.\]

So \(\Psi\) shrinks the distance between \(D\) and \(D^*\). As a result, \(\Psi\) cannot have two distinct fixed points, and the iteration of \(\Psi\) leads to the unique fixed point.

7.2 Monte Carlo experiment and bootstrapped standard errors

The model in this paper deviates from the standard framework for estimating differentiated goods markets by (i) introducing a fixed-point notion for the demand function and (ii) tying together all the city-pair markets by a network structure. The goal of the Monte Carlo experiment here is to check whether the estimator is still well-behaved under such conditions. Fix a network and a set of “true” parameter values. The Monte Carlo experiment first draws a set of the unobservables, \(\xi\) and \(\omega\), then computes a Nash-Bertrand equilibrium to produce a simulated data, and finally recovers the parameter values from the simulated data. The parameter recovery uses the estimator outlined in Section 3. I repeat the experiment many times to obtain a sample of recovered values for each parameter.

The model does not impose any distributional assumption on \(\xi\) and \(\omega\). However, for Monte Carlo experiment, such a distribution needs to be specified. One possibility is to use i.i.d. normal distributions. However, it is reasonable to expect a positive correlation between \(\xi_j\) and \(\xi_k\) if \(j\) and \(k\) belong to the same market. It is also reasonable to expect a positive correlation between \(\omega_j\) and \(\omega_k\) if \(j\) and \(k\) share a link. In fact, these correlations are present in the backed-out values of \(\xi\) and \(\omega\). To replicate the data generating process as much as possible, I add these correlations in the Monte Carlo experiment.

As to the choice of the “true” parameter values, I use the benchmark point estimates in Table 4 and 5. As to the choice of network, it may be ideal to simply use the observed network in the data. However, it is very costly to compute the equilibrium for such a large network and much more costly to repeat the experiment for many times. In principle, there are many alternative choices for the network; one may simulate a random network and then fix it throughout the experiment. To mimic the observed network structure as much as possible, I choose to use a downsized version of the observed network. Specifically, I exclude a link if the observed number of passengers on the link is no more than 2.5% of the size of the associated city-pair market. The resulting “backbone” network has 602 links and \(n^b = 7,353\) routes across all the carriers. For a network of this size, it is possible to compute \(\partial D(\xi, p) / \partial p\) with implicit function theorem (see Section 3.1), which significantly speeds up the computation. The downsized network effectively reduces the data size used to recover the parameters, which should inflate the variances of the estimator. To compensate, I set the standard deviations of the distributions for \(\xi\) and \(\omega\) at \(\sqrt{n^b/n}\) of their levels in the real data.
Notes: The experiment is repeated for 500 times. Each plot shows the kernel density (with optimal bandwidth) of the recovered values for a particular parameter. The vertical line shows the “true” parameter value.

Figure 6: Monte Carlo Results
Figure 6 illustrates the results by showing the kernel densities of the recovered values for four parameters: $\theta_1$, $\theta_2$, the intercept in $\gamma$, and the coefficient in $\gamma$ for the point-to-point distance. Keep in mind that the estimator is sequential (i.e., it first recovers the demand parameters then supply parameters), so the dispersions in $\gamma$ partly come from the estimation errors on the demand parameters. It is seen that each density is centered around the true parameter value and well-behaved. The same result also applies to the other parameters not shown in Figure 6. The standard deviations of the recovered values are used as bootstrapped standard errors, reported in Table 4, 5, and 6.

7.3 Equilibrium computation

Below shows the algorithm that I use to compute a Bertrand-Nash equilibrium. The algorithm basically solves for the prices that satisfy the first-order conditions (2.8). The inputs include the network structure, $\xi$, and marginal costs $mc$. The outputs include the equilibrium prices, demand, and densities.

1. Choose a convergence criterion $\tau > 0$ and an initial value for the price vector $p^\ast$.
2. Iterate $\Psi(p^\ast, \xi, \cdot)$ to find its fixed point $D(p^\ast, \xi)$.
3. Compute $\partial D(p^\ast, \xi)/\partial p$ and $\Delta$ defined as in (3.2). Let $p^{**} = mc - \Delta^{-1}D(p^\ast, \xi)$.
4. If $\|p^{**} - p^\ast\|_\infty < \tau$, exit; otherwise, update $p^\ast$ by moving it closer to $p^{**}$.
5. Repeat step 2-5.

The most time-consuming part of the algorithm is step 3. There are mainly two complexities. First, this step needs to differentiate an implicitly defined demand function w.r.t. tens of thousands of prices; the differentiation w.r.t. each price requires fixed-point iterations of $\Psi$ (The iterations may be avoided for smaller-size networks, see Section 3.1). Second, $\Delta^{-1}D(p^\ast, \xi)$ involves solving a large linear system. (As noted in Section 3.1, $\Delta$ can be organized into a block-diagonal matrix, but the blocks corresponding to the major carriers are still of significant sizes.)

Given the computational complexity, it is important to reduce error propagation. In my experience, it is particularly important to keep the numerical errors small at two places. First, I use a very stringent stop criterion for the fixed point iteration of $\Psi$. Second, I use the two-sided formula (also called the central difference method) when computing the derivative $\partial D(p^\ast, \xi)/\partial p$. 

31
References


